

# Calculus on random integral mappings $I_{(a,b]}^{h,r}$ and their domains\*

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**Abstract.** It is proved that the random integral mappings (some type of functionals of Lévy processes) are always isomorphisms between convolution semigroups of infinitely divisible measures. However, the inverse mappings are no longer of the random integral form. Domains are characterized in many ways. Compositions (iterated integrals) can be expressed as a single random integral mapping. Finally, all obtained results are illustrated by examples.

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*Abbreviated title:* *Calculus on random integral mappings*

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In Jurek-Vervaat (1983) it was proved that in order that

$$a_n(\xi_1 + \xi_2 + \dots + \xi_n) + x_n \Rightarrow \mu, \quad (\star)$$

for some an infinitesimal triangular array  $a_n\xi_j$ ,  $1 \leq j \leq n$ ;  $n \geq 1$ , it is necessary and sufficient that

$$\mu = \mathcal{L}\left(\int_{(0,\infty)} e^{-t} dY_\nu(t)\right), \quad (\star\star)$$

where the random integral is taken with respect to some Lévy process  $(Y_\nu(t), 0 \leq t < \infty)$  such that  $\nu = \mathcal{L}(Y_\nu(1))$  has a logarithmic moment. The expression  $(\star\star)$  was called *a random integral representation* of the measure  $\mu$  and  $Y_\nu$  was referred to as *the background driving Lévy process* (BDLP) of  $\mu$ . Later on, the phenomena of identifying a class limit laws with a collection of laws of random integrals was proved for many other limiting schemes. Still later on, many new classes of probability distributions were simply defined as laws of some random integrals analogous to  $(\star\star)$  without any reference to limiting procedures as  $(\star)$ ; cf. Jurek (2011) (an invited lecture at 10<sup>th</sup> Vilnius Conference) for more details and references; also comp. *The Conjecture* at [www.math.uni.wroc.pl/~zjjurek](http://www.math.uni.wroc.pl/~zjjurek)

In this paper we propose a quite general approach to random integral representations and mappings. For a more convenient way of navigating the body of research in random representations and an ease of comparing results of different Authors we propose here a new and unified way of definitions and notations.

Finally, we will utilize here the notion and properties of image measures, in particular, images under the tensor product of functions. Although our results and proofs are given in the generality of measures on Banach spaces, they can be read on  $\mathbb{R}^d$  without essential knowledge of the functional analysis.

Last but not least, the proposed calculus on random integral mappings and their domains might be formally viewed as an analogue of the calculus on linear operators on Banach spaces and their domains (in functional analysis).

## 0. Notations and brief descriptions of results.

### 0.1. Notations and basic facts

$E$  is a real separable Banach space;

$E'$  is the topological dual of  $E$ ;

$\langle \cdot, \cdot \rangle$  is the dual pair (scalar product) between  $E'$  and  $E$ ;

$\Rightarrow$  denotes the weak convergence of probability measures;

$\mathcal{L}(X)$  is the probability distribution of random variable  $X$ ;

$\stackrel{d}{=}$  means equality in distribution;

$(Y_\nu(t), 0 \leq t < \infty)$ , denote a Lévy process such that  $\mathcal{L}(Y_\nu(1)) = \nu$ ;  
 $D_F[a, b]$  denotes the Skorochod space of  $F$ -valued functions that are right continuous on  $[a, b)$  with left-hand limits on  $(a, b]$ ; in short: *cadlag* functions;  $F$  is a complete separable metric space;  
 $ID(E)$  denotes the set of all infinitely divisible Borel measures on  $E$ ;  
 $\hat{\mu}$  is the characteristic functional (Fourier transform) of  $\mu$ ;  
 $\Phi(y) = \log \hat{\mu}(y)$  is the *Lévy exponent* of  $\mu \in ID(E)$ , ( $y \in E'$ );  
 $\Phi(y) = i < y, z_0 > -\frac{1}{2} < y, Ry > + \int_{E \setminus \{0\}} (e^{i < y, x >} - 1 - i < y, x > 1_B(x)) M(dx)$  (*Lévy-Khintchine formula*);  
 $(z_0 \in E, R$  is a Gaussian covariance operator,  $M$  is a Lévy (spectral) measure and  $B$  is the unit ball in  $E$ );  
 $\nu = [z_0, R, M]$  means  $\nu \in ID$  with the triple from its Lévy-Khintchine formula;  
 $\nu^{*c} = [z_0, R, M]^{*c} := [c \cdot z_0, c \cdot R, c \cdot M]$  for  $c > 0$ ;  
 $(f\mu)$  denotes the image of a measure  $\mu$  under a measurable mapping  $f$ ;  
 $(f \otimes g)(s_1, s_2) := f(s_1) \cdot g(s_2)$  (tensor product) for  $(s_1, s_2) \in S \times S$  and  $f, g : S \rightarrow \mathbb{R}$ ;  
 $I_{(a,b]}^{h,r}$  is the random integral mapping with a space transform function  $h$ , a deterministic monotone time change  $r$  (an inner clock) and the time interval  $(a, b]$ ; cf. (1) below;  
 $\mathcal{D}_{(a,b]}^{h,r}$  the domain of the integral mapping  $I_{(a,b]}^{h,r}$ ;  
 $\mathcal{R}_{(a,b]}^{h,r} := I_{(a,b]}^{h,r}(\mathcal{D}_{(a,b]}^{h,r}) \subset ID$  denotes the range of the mapping  $I_{(a,b]}^{h,r}$ ;  
 $I_{(a,\infty)}^{h,r}(\nu)$  means weak limit of  $I_{(a,b]}^{h,r}(\nu)$  as  $b \rightarrow \infty$ ;  
 $\Phi_{(a,b]}^{h,r}(y) := \log(\widehat{I_{(a,b]}^{h,r}(\nu)})(y)$  when  $\Phi = \log \hat{\nu}(y)$   
 $I_{(a,b]}^{h,r}([z, R, M]) =: [z_{(a,b]}^{h,r}, R_{(a,b]}^{h,r}, M_{(a,b]}^{(h,r)}]$ , cf. (16);

## 0.2. Summary of results

In section 1 the random integral and its basic properties are given. In Theorem 1, in section 2, we prove that mappings  $I_{(a,b]}^{h,r}$  are isomorphisms of some convolution semigroups in  $ID(E)$ . An alternative proof on  $\mathbb{R}^d$  is given in Proposition 2. Subsection 2.5 explores the Mellin transforms as yet another possibly tool for a proof. Then, in section 3, we discuss the domains  $\mathcal{D}_{(a,b]}^{h,r}$  of mappings  $I_{(a,b]}^{h,r}$  (Propositions 3 - 5). In section 3, Theorem 2 shows that all compositions of  $I_{(a,b]}^{h,r}$  (iterated integral mappings) can be expressed as a single integral random mapping. Here the language of tensor products and the notion of image measures are very convenient. In section 4, the inverse mappings to  $I_{(a,b]}^{h,r}$  are discussed in Theorem 3. However, they are no longer of the form  $I_{(a,b]}^{h,r}$ . Section 5, in particular Proposition 6, is devoted

to fixed points of mappings  $I_{(a,b]}^{h,r}$  and to the role of stable distributions. In section 7 the factorization property of measures is discussed (Proposition 8). As a consequence we get that some selfdecomposable (in other words class L distributions) measures have the factorization property. In last section (section 8) we illustrate our results on some new or previously studied integral mappings and semigroups.

## 1. Definitions of random integral mappings.

### 1.1. Integrals on finite intervals

For an interval  $(a, b]$  in a positive half-line, a real-valued continuous bound variation function  $h$  on  $[a, b]$ , a positive monotone right-continuous time change function  $r$  on  $[a, b]$  and a cadlag Lévy stochastic processes  $(Y_\nu(t), 0 \leq t < \infty)$ , let us define via a formal integration by parts formula the following *random integral*

$$\int_{(a,b]} h(t) dY_\nu(r(t)) := h(b)Y_\nu(r(b)) - h(a)Y_\nu(r(a)) - \int_{(a,b]} Y_\nu(r(t)-) dh(t) \in E,$$

and the corresponding *random integral mapping*

$$I_{(a,b]}^{h,r}(\nu) := \mathcal{L}\left(\int_{(a,b]} h(t) dY_\nu(r(t))\right) \in ID, \quad (1)$$

with  $\nu$  in its *domain*  $\mathcal{D}_{(a,b]}^{h,r}$  being a subset of the class ID consisting of those measures  $\nu$  for which the integral (1) is well defined.

Since Lévy processes are semi-martingales the random integral (1) can be defined as an Ito stochastic integral. However, for our purposes we do not need that generality.

If  $\nu \in \mathcal{D}_{(a,b]}^{h,r}$  and  $I_{(a,b]}^{h,r}(\nu)$  have the Lévy exponents  $\Phi$  and  $\Phi_{(a,b]}^{h,r}$ , respectively then, as in Lemma 1.1 in Jurek-Vervaat(1983), we get

$$\Phi_{(a,b]}^{h,r}(y) = \int_{(a,b]} \Phi(h(t)y) dr(t), \quad y \in E' \quad (\text{for nondecreasing } r). \quad (2)$$

Similarly we have that

$$\Phi_{(a,b]}^{h,r}(y) = \int_{(a,b]} \Phi(-h(t)y) |dr(t)|, \quad y \in E' \quad (\text{for decreasing } r), \quad (3)$$

because for  $0 < u < w$ , we have  $\mathcal{L}(Y_\nu(u) - Y_\nu(w)) = (\nu^-)^{*(w-u)}$  where  $\nu^- := \mathcal{L}(-Y_\nu(1))$ . In other words,  $(-Y_\nu(t), t \geq 0) \stackrel{d}{=} (Y_{\nu^-}(t), t \geq 0)$ .

### 1.2. Improper random integrals and invariance property

Integrals over intervals  $(a, b)$  or  $(a, \infty)$  or  $[a, b]$  and others are defined as appropriate weak limits of integrals over intervals  $(a, b]$  in (1). Similarly, we use appropriate limits of (1) when  $|h(a)| = \infty$  or  $r(a) = \infty$ .

Further note that, in fact,  $I_{(a,b]}^{h,r}(\nu)$  is a functional of whole Lévy process on  $(a, b]$ , not only a function of the measure  $\nu$ . Consequently, if for two Lévy processes  $\bar{Y}_\nu$  and  $Y_\nu$  we have that  $(\bar{Y}_\nu(t) : t \geq 0) \stackrel{d}{=} (Y_\nu(t) : t \geq 0)$  then

$$\mathcal{L}\left(\int_{(a,b]} h(t) dY_\nu(r(t))\right) = \mathcal{L}\left(\int_{(a,b]} h(t) d\bar{Y}_\nu(r(t))\right). \quad (1a)$$

### 1.3. Different graphic notations

Note that we have

$$I_{(a,b]}^{h,r}(\nu) = \mathcal{L}\left(\int_0^\infty 1_{(a,b]}(r^*(s)) h(r^*(s)) dY_\nu(s)\right) = I_{(0,\infty)}^{\tilde{h}(s), s}(\nu) \equiv \tilde{I}^h(\nu), \quad (1b)$$

where  $\tilde{h}(s) := 1_{(a,b]}(r^*(s)) h(r^*(s))$  and  $r^*$  is the inverse function of  $r$ .

However, instead of that graphically simpler notation (1b), for a greater flexibility of our considerations we will keep the three parameters: the time interval  $(a, b]$ , the space-value normalization  $h$  and the inner time change  $r$  in symbols and notions related to the random integral mappings (1).

For improper random integrals with decreasing  $r$  with  $r(a+) < \infty$  we have

$$I_{(a,b]}^{h,r}(\nu) = I_{(a,b]}^{-h, r(a+)-r}(\nu) = I_{(a,b]}^{h, r(a+)-r}(\nu^-),$$

that is,

$$\int_{(a,b]} h(t) dY_\nu(r(t)) \stackrel{d}{=} \int_{(a,b]} h(t) dY_{\nu^-}(r(a+) - r(t)),$$

and  $t \rightarrow r(a+) - r(t)$  is a positive increasing function.

### 1.4. Trivial mappings

If  $h(t) \equiv 0$  or  $r(t) \equiv r_0$  (constant function) then

$$\int_{(a,b]} 0 dY_\nu(r(t)) = 0 = \int_{(a,b]} h(t) dY_\nu(r_0) \quad \text{for all } \nu \in ID,$$

are the zero mappings;  $I_{(a,b]}^{0,r}(\nu) = I_{(a,b]}^{h,r_0}(\nu) = \delta_0$ . To exclude that trivial case, in sequel we tacitly assume that  $h$  is not zero function and  $r$  is not constant

function. Equivalently, the three parameters: an interval  $(a, b]$  and functions  $h, r$  satisfy the basic condition

$$0 < \int_{(a,b]} |h(t)| |dr(t)| \quad (4)$$

On the other hand, finiteness of the integral (4) guarantees that degenerate Lévy process  $Y(t) := ta$  (a fixed) can be used as integrators in the integrals (1); cf. formula (14) in Proposition 4 below.

### 1.5. Additivity properties of random mappings

Directly from the definition (1), (2) and (3) we have

$$I_{(a,c]}^{h,r}(\nu) * I_{(c,b]}^{h,r}(\nu) = I_{(a,b]}^{h,r}(\nu), \quad I_{(a,b]}^{h,r_1+r_2}(\nu) = I_{(a,b]}^{h,r_1}(\nu) * I_{(a,b]}^{h,r_2}(\nu),$$

for  $\nu$  in appropriate domains. Note that the first equality is a particular case of the general property that stochastic integrals are independent when the integrands have disjoint supports.

## 2. The mappings $I_{(a,b]}^{h,r}$ are isomorphisms.

### 2.1. Note on a case of improper integrals

The theorem below is valid for other time intervals, in particular for  $(a, \infty)$  that give the improper integrals, with the same proof as we give below for the intervals  $(a, b]$ . Here it is helpful to keep in mind Proposition 1, below and Lemma 1.2 in Jurek-Vervaat (1983).

### 2.2. An isomorphism property

**THEOREM 1.** *For a domain  $\mathcal{D}_{(a,b]}^{h,r}$  of the integral mapping  $I_{(a,b]}^{h,r}$  we have that*

$$I_{(a,b]}^{h,r} : \mathcal{D}_{(a,b]}^{h,r} \rightarrow \mathcal{R}_{(a,b]}^{h,r} := I_{(a,b]}^{h,r}(\mathcal{D}_{(a,b]}^{h,r}) \subset ID \quad (5)$$

*is an isomorphism between the corresponding measure convolution semigroups. Moreover, for any  $s > 0$ ,  $\nu \in \mathcal{D}_{(a,b]}^{h,r}$  if and only if  $\nu^{*s} \in \mathcal{D}_{(a,b]}^{h,r}$  and*

$$I_{(a,b]}^{h,r}(\nu^{*s}) = (I_{(a,b]}^{h,r}(\nu))^{*s} = (I_{(a,b]}^{h,sr}(\nu)).$$

*And for  $u \in \mathbb{R}$  and the dilation operator  $T_u$ ,  $\nu \in \mathcal{D}_{(a,b]}^{h,r}$  if and only if  $T_u \nu \in \mathcal{D}_{(a,b]}^{h,r}$  and*

$$I_{(a,b]}^{h,r}(T_u \nu) = T_u(I_{(a,b]}^{h,r}(\nu)) = I_{(a,b]}^{uh,r}(\nu).$$

*Furthermore, if  $|r(b) - r(a+)| < \infty$  then  $\nu \rightarrow I_{(a,b]}^{h,r}(\nu)$  is a continuous in the weak topology.*

*Proof.* In contrary, suppose that there are two Lévy exponents  $\Phi_1$  and  $\Phi_2$ , with the corresponding triples  $[x_1, R_1, M_1]$  and  $[x_2, R_2, M_2]$ , respectively and such that

$$\int_{(a,b]} (\Phi_1 - \Phi_2)(h(t)y) dr(t) = 0, \quad y \in E'$$

Putting  $R := R_1 - R_2$  and  $m := M_1 - M_2$  (signed measure) and taking the real parts we arrive at equality

$$2^{-1} \int_{(a,b]} h^2(t) dr(t) \langle y, Ry \rangle = \int_{(a,b]} \int_{E \setminus \{0\}} (\cos \langle y, h(t)x \rangle - 1) m(dx) dr(t). \quad (6)$$

Since the above left-hand side satisfies in  $y$  the parallelogram law (i.e.,  $g$  satisfies parallelogram law if  $g(y_1 + y_2) + g(y_1 - y_2) - 2g(y_1) - 2g(y_2) = 0$ ) and  $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$ , therefore for any  $y_1$  and  $y_2$  we get

$$\int_{(a,b]} \int_{E \setminus \{0\}} (1 - \cos \langle y_1, h(t)x \rangle)(1 - \cos \langle y_2, h(t)x \rangle) m(dx) dr(t) = 0 \quad (7)$$

Furthermore, without loss of generality we may assume in (6) that the signed  $m$  is also symmetric one (if necessary we take its symmetrization  $m + m^-$ ). Let us define signed measures

$$\tilde{m}(A) := \int_{(a,b]} (T_{h(s)} m(A)) dr(t) = \int_{(a,b]} \int_E 1_A(h(t)x) m(dx) dr(t) \quad (8)$$

and

$$\tilde{m}_y(A) := \int_A (1 - \cos \langle y, x \rangle) \tilde{m}(dx). \quad (9)$$

Since Lévy spectral measures integrate the functions  $1 \wedge |\langle y, \cdot \rangle|^2$  over  $E$  and  $1 - \cos \langle y, x \rangle \leq 2(1 \wedge |\langle y, x \rangle|^2)$  we conclude that  $\tilde{m}_y$  are finite signed measures and (7) means that

$$\int_{E \setminus \{0\}} (1 - \cos \langle y_1, x \rangle) \tilde{m}_{y_2}(dx) = 0, \quad \text{for all } y_1.$$

Thus

$$\int_E \exp i \langle y_1, x \rangle \tilde{m}_{y_2}(dx) = \tilde{m}_{y_2}(E),$$

and hence  $\tilde{m}_{y_2}$  is concentrated at zero. Since Lévy spectral measures have no atoms at the origin, so  $\tilde{m}_{y_2} \equiv 0$  and  $\tilde{m}_{y_2}(E) = 0$  for all  $y_2$ . Consequently,  $\tilde{m} \equiv 0$  and  $M_1 = M_2$ , provided  $r$  is not constant (degenerate).

If the integral on right-hand side in (6) is zero we get that  $R_1 = R_2$ , which in turn give that  $x_1 = x_2$  (equality of shifts). This completes that the integral mappings  $I_{(a,b]}^{h,r}$  are one-to-one.

The homomorphism property of  $I_{(a,b]}^{h,r}$ , that is,

$$I_{(a,b]}^{h,r}(\nu_1 * \nu_2) = I_{(a,b]}^{h,r}(\nu_1) * I_{(a,b]}^{h,r}(\nu_2),$$

in terms of Lévy exponents follows immediately from (2) or (3).

Similarly, from (2) and (3) we get the other two properties, i.e., closure under convolution powers and transformation under bounded linear operators.

For the continuity, let us note that  $0 \leq |r(b) - r(a)| < \infty$  and the cadlag property imply that functions  $t \rightarrow Y(r(t))$  are bounded and with at most countable many discontinuities; cf. Billingsley (1968), Chapter 3, Lemma 1. Furthermore, the mapping

$$D_E[a, b] \ni y \rightarrow \int_{(a,b]} h(t) dy(r(t)) := h(b)y(r(b)) - h(a)y(r(a)) - \int_{(a,b]} y(r(t)-) dh(t) \in E, \quad (10)$$

is continuous in Skorohod topology (for details see Billingsley (1968), p. 121.). Furthermore, if  $\nu_n \Rightarrow \nu$  then  $(Y_{\nu_n}(t), a \leq t \leq b) \Rightarrow (Y_\nu(t), 0 \leq t \leq b)$  in  $D_E[a, b]$ . Consequently, we have

$$\mathcal{L}\left(\int_{(a,b]} h(t) dY_{\nu_n}(r(t))\right) \Rightarrow \mathcal{L}\left(\int_{(a,b]} h(t) dY_\nu(r(t))\right)$$

which proves the continuity and completes the proof of Theorem 1.

**COROLLARY 1.** *For  $h$  and  $r$  as above and Lévy spectral measures  $M$  and  $N$ , if  $M_{(a,b]}^{h,r} = N_{(a,b]}^{h,r}$  then  $M=N$ .*

Proof is an immediate consequence of Theorem 1.

**Remark 1.** (i) In some specific cases of mappings such as  $I_{(0,\infty)}^{e^{-t},t}$  or  $I_{(0,1]}^{t,t^\beta}$  the one-to-one property is much easier to prove using Fourier or Laplace transforms; cf. Jurek-Vervaat (1983) or Jurek (1988), (2007).

(ii) Let us note that Corollary 1, among others, provides different and more general proof of one-to-one property of arcsine transformations from section 2.2 in Maejima, Perez-Abreu and Sato (2010).

(iii) Continuity of the mappings  $I_{(a,b]}^{h,r}$  in  $\mathbb{R}^d$  easily follows by the characteristic functional argument via the identity (2)



### 2.3. Integral convolution factors.

We say that probability measures  $\mu$  on  $E$  is a *convolution factor* of a measure  $\rho$  if there exists a measure  $\nu$  such that  $\mu * \nu = \rho$ ; in symbols we write  $\mu \prec \rho$ .

**PROPOSITION 1.** *For  $\nu \in \mathcal{D}_{(a,b]}^{h,r}$ , the family  $\{I_{(a,x]}^{h,r}(\nu) : a < x < b\}$  is sequentially shift convergent for  $x \uparrow c \leq b$  or  $x \downarrow c \geq a$ .*

*Proof.* Note that if  $a < x_1 < x_2 < \dots < x_n < \dots \uparrow c \leq b$  then

$$I_{(a,x_1]}^{h,r}(\nu) \prec I_{(a,x_2]}^{h,r}(\nu) \prec \dots \prec I_{(a,b]}^{h,r}(\nu),$$

and by Theorem 5.3 in Parthasarathy (1968) there exist sequence  $\delta_{z_n}$  and a measure  $\rho$  such that  $I_{(a,x_n]}^{h,r}(\nu) * \delta_{z_n} \Rightarrow \rho$  as  $n \rightarrow \infty$ . Similarly we argue in the remaining case.

### 2.4. One-to-one property of $I_{(a,b]}^{h,r}$ via weak limits

Knowing integrals (1) over a family of intervals  $(c, x]$ , as  $x \downarrow c$ , we can retrieve the measure  $\nu$  as is it seen below.

**PROPOSITION 2.** *Assume that  $\nu \in ID(\mathbb{R}^d)$ . If  $h$  is of bounded variation continuous,  $r$  is nondecreasing and there exists  $c \in (a, b)$  such that  $h(c) \neq 0$  and  $r'(c) > 0$  then*

$$\mathcal{L}\left(\int_c^x \frac{h(t)}{h(c)} dY_\nu\left(\frac{r(t)}{r'(c)}\right)\right)^{* \frac{1}{x-c}} \Rightarrow \nu, \quad \text{as } x \downarrow c. \quad (11)$$

*Proof.* The equality above, for measures on  $\mathbb{R}^d$ , in terms of Lévy exponents is equivalent to the following claim

$$\lim_{x \rightarrow c} \frac{1}{x - c} \int_c^x \Phi\left(\frac{h(t)}{h(c)} y\right) \frac{r'(t)}{r'(c)} dt = \Phi(y) \quad \text{for all } y,$$

that is obviously true because of de l'Hospital rule.

**COROLLARY 2.** *For a measure  $\nu \in ID(\mathbb{R}^d)$ ,*

$$\mathcal{L}\left(\int_a^{a+1/n} h(t) dY_\nu(r(t))\right)^{*n} \Rightarrow T_{h(a+)} \nu^{*r'(a+)} \quad \text{as } n \rightarrow \infty.$$

### 2.5. One-to-one property of $I_{(a,b]}^{h,r}$ via Mellin transform

Recall that the *Mellin transform*  $M[f; z]$  of a function  $f$  on positive half-line is defined as

$$M[f; z] := \int_0^\infty f(x)x^{z-1}dx, \text{ provided it exists for } z. \quad (12)$$

Recall that

$$g(z) := \int_0^\infty \frac{1 - \cos x}{x^{z+1}} dx < \infty \text{ for } 0 < z < 2; \quad g(1) = \pi/2.$$

Suppose that the time change has a form  $dr(t) := k(t)dt$ , for some positive function  $k$  and consider  $\Phi$  symmetric and without Gaussian part, that is,  $\Phi(y) = \int_{E \setminus \{0\}} (\cos \langle y, x \rangle - 1) M(dx)$ . Then

$$\tilde{\Phi}(y) \equiv \Phi_{(0,\infty)}^{s,r}(y) = \int_0^\infty \Phi(sy)k(s)ds.$$

Note that for each fixed  $y$ , functions  $t \rightarrow \Phi(ty)$  and  $t \rightarrow \tilde{\Phi}(ty)$ ,  $0 < t < \infty$  satisfy the integrability condition (12) with  $-z \in (0, 2)$ . Then we get

$$\begin{aligned} M[\tilde{\Phi}(\cdot y); \sigma] &= \int_0^\infty \tilde{\Phi}(ty)t^{\sigma-1}dt = \int_0^\infty \left( \int_0^\infty \Phi(sty)k(s)ds \right) t^{\sigma-1}dt \\ &= \int_0^\infty \Phi(wy)w^{\sigma-1}dw \int_0^\infty k(s)s^{-\sigma}ds = M[\Phi(\cdot y); \sigma] M[k; 1 - \sigma], \end{aligned} \quad (13)$$

provided  $M[k; 1 - \sigma]$  exists. Since Mellin transform  $M[f; \sigma]$  uniquely determines its function  $f$  we infer that the integral  $I_{(0,\infty)}^{s,r}(\nu)$  uniquely determines the measure  $\nu$ .

### 3. Domains $\mathcal{D}_{(a,b]}^{h,r}$ of integral mappings $I_{(a,b]}^{h,r}$ .

#### 3.1. Domains on Banach spaces $E$ .

**PROPOSITION 3.** *In order that  $\mathcal{D}_{(a,b]}^{h,r} = ID(E)$  it is necessary and sufficient that integrals  $\int_{(a,b]} y(r(t)-) dh(t)$  exists for all  $y \in D_E[0, \infty)$ .*

*In particular, for an improper random integral if  $|r(b) - r(a+)| < \infty$  then  $\mathcal{D}_{(a,b]}^{h,r} = ID(E)$ .*

*Proof.* Because Lévy processes  $Y$  are cadlag and the random integrals (1) are defined by the formal integration by parts, as in the formula (10), we infer the claim concerning the domains.

If the range of  $r$  is bounded then using the fact that cadlag functions on bounded intervals are integrable (cf. Billingsley (1968), p.121) we get that the integral in (10) is well-defined. This concludes the second claim.

In terms of the Lévy-Khintchine representation domains are characterized as follows:

**PROPOSITION 4.** A measure  $\nu = [z, R, M] \in \mathcal{D}_{(a,b]}^{h,r}$  if and only if

$$\int_{(a,b]} |h(t)| |dr(t)| < \infty, \text{ if } z \neq 0; \quad \int_{(a,b]} h^2(t) |dr(t)| < \infty, \text{ if } R \neq 0, \quad (14)$$

and for the  $\sigma$ -algebra  $\mathcal{B}_0$  of Borel subsets of  $E \setminus \{0\}$ , the set function

$$\mathcal{B}_0 \ni A \rightarrow \int_{(a,b]} [T_{h(t)} M(A)] |dr(t)| \text{ is a Lévy spectral measure on } E. \quad (15)$$

Moreover, if  $I_{(a,b]}^{h,r}(\nu)$  is determined by the triple  $[z_{(a,b]}^{h,r}, R_{(a,b]}^{h,r}, M_{(a,b]}^{(h,r)}]$  and  $r$  is nondecreasing then

$$\begin{aligned} (i) \quad z_{(a,b]}^{h,r} &= \left( \int_{(a,b]} h(t) dr(t) \right) \cdot z + \int_{(a,b]} h(t) \int_{E \setminus \{0\}} [1_B(h(t)x) - 1_B(x)] x M(dx) dr(t); \\ (ii) \quad R_{(a,b]}^{h,r} &= \left( \int_{(a,b]} h(t)^2 dr(t) \right) \cdot R; \\ (iii) \quad M_{(a,b]}^{h,r}(A) &= \int_{(a,b]} [T_{h(t)} M(A)] dr(t) = \int_{(a,b]} \int_{E \setminus \{0\}} 1_A(h(t)x) M(dx) dr(t), \end{aligned} \quad (16)$$

*Proof.* From formulae in Section 0.1, (2) and from the uniqueness of the triple (shift vector, Gaussian covariance and Lévy spectral measure) in the Lévy-Khintchine formula we get the above claims and the three formulae in (16).

**COROLLARY 3.** If  $M_{(a,b]}^{h,r}$  is a Lévy spectral measure on  $E$  then

$$\int_{(a,b]} (1 \wedge h^2(s)) |dr(s)| < \infty.$$

*Proof.* For  $y \in E'$  and the mapping  $\pi_y(x) := \langle y, x \rangle$  ( $x \in E$ ), the image measure  $\pi_y(M_{(a,b]}^{h,r})$  is a Lévy spectral measure on  $\mathbb{R}$ . Since for positive  $s$  and  $t$  we have  $(1 \wedge s)(1 \wedge t) \leq (1 \wedge st)$  therefore we have that

$$\begin{aligned} & \left( \int_{(a,b]} (1 \wedge h^2(s)) |dr(s)| \right) \cdot \left( \int_E (1 \wedge \langle y, x \rangle^2) dM(x) \right) \\ & \leq \int_{(a,b]} \int_E (1 \wedge \langle y, h(s)x \rangle^2) dM(x) |dr(s)| \\ & = \int_E (1 \wedge \langle y, u \rangle^2) M_{(a,b]}^{h,r}(du) = \int_{\mathbb{R}} (1 \wedge w^2) (\pi_y M_{(a,b]}^{h,r})(dw) < \infty, \end{aligned}$$

which concludes the proof.

Here are some sufficient conditions (for a measure) in order to be in a domain:

**PROPOSITION 5.** (i) For  $0 < p \leq 2$ , if the integral  $\int_{(a,b]} |h(t)|^p |dr(t)|$  exists then all symmetric  $p$ -stable measures  $\gamma_p^\circ$  on  $E$  are in the domain  $\mathcal{D}_{(a,b]}^{h,r}$ . (ii) If a positive Borel measure  $N$  on  $E$  integrates the function  $\|x\|$  then  $N$  is a Lévy spectral measure and  $\nu = [z, 0, N]$  has finite first moment. Moreover, if  $\int_{(a,b]} |h(t)| |dr(t)| < \infty$  then  $[z, 0, N] \in \mathcal{D}_{(a,b]}^{h,r}$ .

*Proof.* For (i), recall that Lévy exponents of symmetric  $p$ -stable non Gaussian measures for  $0 < p < 2$  are of the form

$$\Phi(y) \equiv -\log \hat{\gamma}_p^\circ(y) = \int_{\{\|x\|=1\}} | \langle y, x \rangle |^p m(dx)$$

for some finite measure  $m$  on the unit sphere; cf. Araujo and Giné (1980), Chapter III, Theorem 6.16. Hence and from (2)

$$y \rightarrow \int_{(a,b]} \Phi((h(t)y)dr(t) = \int_{(a,b]} |h(t)|^p dr(t) \int_{\{\|x\|=1\}} | \langle y, x \rangle |^p m(dx)$$

is also a Lévy exponent and thus the random integral is well defined. The case of symmetric Gaussian ( $p = 2$ ) follows from Corollary 4 (ii).

For part (ii), since  $\int_{E \setminus \{0\}} (1 \wedge \|x\|) N(dx) < \infty$ , therefore  $N$  is a Lévy measure by Araujo-Giné (1980), Chapter 3, Theorem 6.3. Since also  $\int_{(\|x\|>1)} \|x\| N(dx) < \infty$  we conclude that  $\nu$  has finite first moment. Furthermore for measure  $N_{(a,b]}^{h,r}$  given by (14) we have

$$\begin{aligned} \int_{E \setminus \{0\}} (1 \wedge \|x\|) N_{(a,b]}^{h,r}(dx) &= \int_{(a,b]} \int_{E \setminus \{0\}} (1 \wedge |h(t)|\|x\|) N(dx) |dr(t)| \\ &\leq \left( \int_{(a,b]} |h(t)| |dr(t)| \right) \left( \int_E \|x\| N(dx) \right) < \infty, \quad (17) \end{aligned}$$

and again by Theorem 6.3 in Chapter 3 in Araujo-Giné (1980) we conclude that  $N_{(a,b]}^{h,r}$  is a Lévy spectral measure. Thus  $\nu \in \mathcal{D}_{(a,b]}^{h,r}$  and the proof is completed.

### 3.2. Domains on Hilbert space $H$

On real separable Hilbert spaces we have complete characterization of covariance operators and more importantly, for the considerations here, we know that

$$M \text{ is a Lévy measure on } H \text{ iff } M\{0\} = 0 \text{ and } \int_H (1 \wedge \|x\|^2) M(dx) < \infty,$$

cf. Parthasarathy (1968), Chapter VI. With the above and Proposition 4 we have

**COROLLARY 4.** A measure  $\nu = [z, R, M] \in \mathcal{D}_{(a,b]}^{h,r}(H)$  if and only if

- (i)  $\int_{(a,b]} |h(t)| |dr(t)| < \infty$ , provided  $z \neq 0$ ,
- (ii)  $\int_{(a,b]} h^2(t) |dr(t)| < \infty$ , provided  $R \neq 0$ ,
- (iii)  $\int_{(a,b]} \int_{E \setminus \{0\}} (1 \wedge |h(t)|^2 ||x||^2) M(dx) |dr(t)| < \infty$ , provided  $M \neq 0$ .

**Remark 2.** Since for all positive  $s$  and  $t$ ,  $(1 \wedge s)(1 \wedge t) \leq (1 \wedge st)$ , from the above condition (iii) we infer that if  $M_{(a,b]}^{h,r}$  is a Lévy spectral measure (on  $H$ ) then so is  $M$  and

$$\int_{(a,b]} (1 \wedge |h(t)|^2) |dr(t)| < \infty.$$

Thus it is a necessary condition for the triple: an interval  $(a, b]$  and functions  $h$  and  $r$ , whenever  $M \neq 0$ .

**PROPOSITION 6.** For triples  $(a, b], h$  and  $r$  satisfying the conditions (i) and (ii) from Corollary 4, all infinitely divisible measures with finite second are in their domains, that is,  $ID_2(H) \subset \mathcal{D}_{(a,b]}^{h,r}$ , for arbitrary Hilbert space  $H$ .

*Proof.* In view of Jurek-Smalara (1981) or Proposition 1.18.13 in Jurek-Mason(1993) or Theorem 25.3 in Sato (1999) we know that  $\nu = [z, R, M] \in ID_2(H)$  if and only if  $\int_{(||x||>1)} ||x||^2 M(dx) < \infty$ . Since

$$\int_{(a,b]} \int_H (1 \wedge |h(t)|^2 ||x||^2) M(dx) |dr(t)| \leq \int_{(a,b]} h^2(t) |dr(t)| \int_H ||x||^2 M(dx) < \infty,$$

( on  $H$ , Lévy measure  $M$  always integrates  $||x||^2$  in the unit ball), we conclude that  $\nu \in \mathcal{D}_{(a,b]}^{h,r}$ , which completes the proof.

#### 4. Compositions of random integral mappings $I_{(a,b]}^{h,r}$

##### 4.1. Equivalent mappings

We say that two integral mappings  $I_{(a,b]}^{h,r}$  and  $I_{(a_1,b_1]}^{h_1,r_1}$  are *equivalent* if

$$\mathcal{D}_{(a,b]}^{h,r} = \mathcal{D}_{(a_1,b_1]}^{h_1,r_1} \quad \text{and} \quad I_{(a,b]}^{h,r}(\mathcal{D}_{(a,b]}^{h,r}) = I_{(a_1,b_1]}^{h_1,r_1}(\mathcal{D}_{(a_1,b_1]}^{h_1,r_1}), \quad (18)$$

and we write  $I_{(a,b]}^{h,r} = I_{(a_1,b_1]}^{h_1,r_1}$ . In terms of Lévy exponents the above means that

$$\int_{(a_1,b_1]} \Phi(h_1(t)y) dr_1(t) = \int_{(a_2,b_2]} \Phi(h_2(t)y) dr_2(t), \quad \text{for all } y \in E'$$

for Lévy exponents  $\Phi$  (measures) in appropriate domains.

**Remark 3.** Mappings  $I_{(0,\infty)}^{e^{-t},t}$  and  $I_{(0,1)}^{s,-\log s}$  are equivalent. Similarly,  $I_{(0,1]}^{t,t^\beta}$  and  $I_{(0,1]}^{t^{1/\beta},t}$ , for  $\beta > 0$ . This follows from above without specifying the domains.

#### 4.2. Iterated random integral mappings

Below let the time change  $r(t)$ ,  $a < t \leq b$ , be either  $\rho\{s : s > t\}$  or  $\rho\{s : s \leq t\}$  for some positive, possibly infinite, measure  $\rho$  on a positive half-line.

For functions  $h_1, \dots, h_m$ , intervals  $(a_1, b_1], \dots, (a_m, b_m]$  and measures  $\rho_1, \dots, \rho_m$  let us define

$$\begin{aligned} \mathbf{h} &:= h_1 \otimes \dots \otimes h_m, \quad (\text{tensor product of functions}) \\ \text{i.e. } \mathbf{h}(t_1, t_2, \dots, t_m) &:= h_1(t_1) \cdot h_2(t_2) \cdot \dots \cdot h_m(t_m), \text{ where } a_i < t_i \leq b_i; \\ (\mathbf{a}, \mathbf{b}] &:= (a_1, b_1] \times \dots \times (a_m, b_m], \quad \boldsymbol{\rho} := \rho_1 \times \dots \times \rho_m \quad (\text{product measure}) \end{aligned} \quad (19)$$

**THEOREM 2.** Let functions  $h_i$ , measures  $\rho_i$  (given by increments of functions  $r_i$ ) and intervals  $(a_i, b_i]$ , for  $i = 1, 2, \dots, m$ , be as above.

If the image  $\mathbf{h}((\mathbf{a}, \mathbf{b}]) = (c, d] \subset \mathbb{R}^+$  and  $\nu \in ID(E)$  is from an appropriate domain then we have

$$I_{(a_1, b_1]}^{h_1, \rho_1} (I_{(a_2, b_2]}^{h_2, \rho_2} (\dots (I_{(a_m, b_m]}^{h_m, \rho_m} (\nu)))) = I_{(c, d]}^{t, \mathbf{h} \boldsymbol{\rho}} (\nu) \quad (20)$$

where  $\mathbf{h} \boldsymbol{\rho}$  is the image of the product measure  $\boldsymbol{\rho} = \rho_1 \times \dots \times \rho_m$  under the mapping  $\mathbf{h} := h_1 \otimes \dots \otimes h_m$ .

*Proof.* For  $\nu \in \mathcal{D}_{(a, b]}^{h, r}$  and its Lévy exponent  $\Phi$  let us define the (script) mapping  $\mathcal{I}_{(a, b]}^{h, r}$  as follows

$$\mathcal{I}_{(a, b]}^{h, r}(\Phi)(y) := \Phi_{(a, b]}^{h, r} = \int_{(a, b]} \Phi(\pm h(s)y) d(\pm)r(s), \quad (21)$$

where the sign minus is in the case of decreasing time change  $r$ . Then to justify (20) it is enough to notice that

$$\begin{aligned} &\mathcal{I}_{(a_1, b_1]}^{h_1, \rho_1} (\mathcal{I}_{(a_2, b_2]}^{h_2, \rho_2} (\dots (\mathcal{I}_{(a_m, b_m]}^{h_m, \rho_m} (\Phi))))(y) \\ &= \int_{(a_1, b_1]} \int_{(a_1, b_2]} \dots \int_{(a_m, b_m]} \Phi(h_1(t_1) h_2(t_2) \dots h_m(t_m) y)) dr_m(t_m) \dots dr_2(t_2) dr_1(t_1) \\ &= \int_{(\mathbf{a}, \mathbf{b}]} \Phi(h_1 \otimes \dots \otimes h_m(s) y) \boldsymbol{\rho}(ds) = \int_{(c, d]} \Phi(t y) (\mathbf{h} \boldsymbol{\rho})(dt), \end{aligned} \quad (22)$$

which follows from the Fubini and the image measure theorems.

In view of the definitions of the tensor product and the product measures we have

$$h_1 \otimes \dots \otimes h_m (\rho_1 \times \rho_2 \times \dots \times \rho_m) = h_{\sigma(1)} \otimes \dots \otimes h_{\sigma(m)} (\rho_{\sigma(1)} \times \rho_{\sigma(2)} \times \dots \times \rho_{\sigma(m)})$$

for any permutation  $\sigma$  of  $1, 2, \dots, m$ . Hence

**COROLLARY 5.** *Random integrals  $I_{(a_i, b_i]}^{h_i, \rho_i}$ ,  $i = 1, 2, \dots, m$ , commute on the domain  $\mathcal{D}_{(c, d]}^{t, \mathbf{h} \rho}$ , where  $\rho = \rho_1 \times \dots \times \rho_m$  and  $\mathbf{h} := h_1 \otimes \dots \otimes h_m$ .*

In case of probability measures  $\rho_i$ , the time change function  $r$  is a cumulative probability distribution and we have

**COROLLARY 6.** *Let assume that  $r_i(t) := \rho_i(\{s \in (a_i, b_i] : a < s \leq t\})$ , where  $\rho_i$  are probability measure on  $(a_i, b_i]$  that are distributions of random variables  $Z_i$ , for  $1 \leq i \leq m$ . If  $Z_1, Z_2, \dots, Z_m$  are stochastically independent then*

$$r(t) := \mathbf{h} \rho(s \leq t) = P[h_1(Z_1) \cdot \dots \cdot h_m(Z_m) \leq t].$$

The above we can apply, for instance, to  $h_i(t) := |t|$  on positive half-line and independent standard normal variable  $Z_i$ . That case was investigated in  $\mathbb{R}^d$  by Aoyama (2009) via polar decomposition of Lévy spectral measures.

#### 4.3. Inclusion of ranges of integral mappings

If a random mapping is a composition of other mappings we may infer some inclusions of their ranges. Namely we have

**COROLLARY 7.** *If an equality  $I_{(a, b]}^{h, r} = I_{(a_1, b_1]}^{h_1, r_1} \circ I_{(a_2, b_2]}^{h_2, r_2}$  (a composition) holds on the domain  $\mathcal{D}_{(a, b]}^{h, r}$  then we have*

$$\mathcal{R}_{(a, b]}^{h, r} \equiv I_{(a, b]}^{h, r}(\mathcal{D}_{(a, b]}^{h, r}) \subset I_{(a_1, b_1]}^{h_1, r_1}(\mathcal{D}_{(a_1, b_1]}^{h_1, r_1}) \cap I_{(a_2, b_2]}^{h_2, r_2}(\mathcal{D}_{(a_2, b_2]}^{h_2, r_2}) = \mathcal{R}_{(a_1, b_1]}^{h_1, r_1} \cap \mathcal{R}_{(a_2, b_2]}^{h_2, r_2}$$

*Proof.* From the equality of the above mappings we get

$$I_{(a, b]}^{h, r}(\mathcal{D}_{(a, b]}^{h, r}) = I_{(a_1, b_1]}^{h_1, r_1}(I_{(a_2, b_2]}^{h_2, r_2}(\mathcal{D}_{(a, b]}^{h, r})) \text{ and hence } I_{(a_2, b_2]}^{h_2, r_2}(\mathcal{D}_{(a, b]}^{h, r}) \subset \mathcal{D}_{(a_1, b_1]}^{h_1, r_1}.$$

Therefore  $I_{(a, b]}^{h, r}(\mathcal{D}_{(a, b]}^{h, r}) \subset I_{(a_1, b_1]}^{h_1, r_1}(\mathcal{D}_{(a_1, b_1]}^{h_1, r_1})$ . Because of the commutativity we get  $I_{(a, b]}^{h, r}(\mathcal{D}_{(a, b]}^{h, r}) \subset I_{(a_2, b_2]}^{h_2, r_2}(\mathcal{D}_{(a_2, b_2]}^{h_2, r_2})$ , which completes a proof.

#### 4.4. An example of application of Theorem 2

**LEMMA 1.** *Let  $h_1(t) := e^{-t}$ ,  $r_1(t) := t$ ,  $h_2(s) := s$  and  $r_2(s) := 1 - e^{-s}$ . Then the corresponding measures are:  $d\rho_1(t) = dt$ ,  $d\rho_2(s) = e^{-s}ds$  and  $d\mathbf{h} \rho(t, s) = d(\rho_1 \times \rho_2)(t, s) = e^{-s} dt ds$ . Finally, for the image measure  $\mathbf{h} \rho(dw) = (h_1 \otimes h_2)(\rho_1 \times \rho_2)(dw) = \frac{e^{-w}}{w} dw$ .*

*Proof.* For Borel measurable, bounded and non-negative functions  $g$  we have

$$\begin{aligned} \int_0^\infty g(u)(h_1 \otimes h_2)(\rho_1 \times \rho_2)(du) &= \int_0^\infty \int_0^\infty g((h_1 \otimes h_2)(t, s))\rho_1(dt)\rho_2(ds) \\ &= \int_0^\infty \int_0^\infty g(e^{-t}s)dt e^{-s}ds = \int_0^\infty \left( \int_0^s g(w)\frac{1}{w}dw \right) e^{-s}ds = \int_0^\infty g(s)\frac{e^{-s}}{s}ds, \end{aligned}$$

which completes the proof of Lemma 1.

From Theorem 2, Corollary 5 and Lemma 1 we conclude that

**COROLLARY 8.** *For  $\nu \in ID_{\log}$  we have*

$$I_{(0,\infty)}^{t, 1-e^{-t}}(I_{(0,\infty)}^{e^{-s}, s}(\nu)) = I_{(0,\infty)}^{e^{-s}, s}(I_{(0,\infty)}^{t, 1-e^{-t}}(\nu)) = I_{(0,\infty)}^{-w, \Gamma(0;w)}(\nu) = I_{(0,\infty)}^{w, \Gamma(0;w)}(\nu^-)$$

Moreover,  $\Gamma(0; w) = (h_1 \otimes h_2)(\rho_1 \times \rho_2)(\{x : x > w\}) = \int_w^\infty \frac{e^{-s}}{s} ds$  for  $w > 0$ .

**Remark 4.** (a) For the Euler constant  $\mathbf{C}$  we have

$$-\Gamma(0; w) = Ei(-w) = \mathbf{C} + \ln w + \int_0^w \frac{e^{-t} - 1}{t} dt, \text{ for } w > 0,$$

where  $Ei$  is the special *exponential-integral* function; cf. Gradshteyn-Ryzhik (1994), formulae 8.211 and 8.212.

(b) Recall that the class of integrals  $I_{(0,\infty)}^{t, 1-e^{-t}}(ID) \equiv \mathcal{E}$  was introduced in Jurek (2007), where the mapping  $I_{(0,\infty)}^{t, 1-e^{-t}}$  was denoted by  $\mathcal{K}^{(e)}$ ; ((e) for exponential cumulative distribution function). More importantly, the class  $\mathcal{E}$  was related to the class of Voiculescu  $\boxplus$  free-infinitely divisible measures; cf. Corollary 6 in Jurek (2007). Note also that  $I_{(0,\infty)}^{t, 1-e^{-t}} = I_{(0,1]}^{-\log s, s}$  and thus it coincides with the upsilon mapping  $\Upsilon$  studied in Barndorff-Nielsen, Maejima and Sato (2006).

(c) Similarly  $I_{(0,\infty)}^{e^{-s}, s}(ID_{\log}) \equiv L$  coincides with the Lévy class of selfdecomposable probability measures; cf. Jurek-Vervaat (1983), Theorem 3.2.

(d) Finally we get identity  $I_{(0,\infty)}^{e^{-s}, s}(I_{(0,\infty)}^{t, 1-e^{-t}}(ID_{\log})) \equiv T$ , which is the Thorin class; cf. Grigelionis (2007), Maejima and Sato (2009) or Jurek (2011).

From Corollary 7 and Remark 4(d) we infer that

**COROLLARY 9.** *For the three classes: Thorin class  $T$ , Lévy class  $L$  (self-decomposable measures) and  $\mathcal{E}$  we have that  $T \subset L \cap \mathcal{E}$*



(These inclusions, on  $\mathbb{R}^d$ , were first noticed in Barndorff-Nielsen, Maejima and Sato (2006) and also in Remark 2.3 in Maejima-Sato (2009) but using completely different methods.)

## 5. Identity and inverse integral mappings.

### 5.1. Identity random integral mappings

Note that whenever  $0 < r_0 := |r(b) - r(a+)| < \infty$  and  $h(t) \equiv 1$  (constant) then

$$I_{(a,b]}^{1,r(t)/r_0}(\nu) = \nu \quad \text{for all } \nu \in ID. \quad (23)$$

So all mappings  $I_{(a,b]}^{1,r(t)/r_0}$  play a role of the neutral element, under the composition, in the family of all integral mappings. In fact, for  $r(\cdot)/r_0$  one may take any time change whose increment over the interval  $(a, b]$  is equal to 1.

Similarly, if  $\delta_u(t) := 1_{[u,\infty)}(t)$ ,  $h(u) \neq 0$  ( $u$  is fixed) and  $u \in (a, b]$  then from (1) or (2) we have

$$I_{(a,b]}^{h/h(u),\delta_u}(\nu) = \nu \quad (24)$$

and thus they also play the role of neutral mappings.

In the first instance, i.e., (23), for the time change one may take any strictly monotone function while the space change  $h$  is trivial. In the second case, i.e., (24), the space change is quite arbitrary but time change  $r$  is one point jump function.

Integrals (23) and (24) are equivalent and will be called *identity random integral mappings*.

### 5.2. An inverse of a random integral mapping

In view of Theorem 1, there exists the inverse mapping  $(I_{(a,b]}^{h,r})^{-1}$ . In fact we have

**THEOREM 3.** *The mapping  $(I_{(a,b]}^{h,r})^{-1} : \mathcal{R}_{(a,b]}^{h,r} \equiv (I_{(a,b]}^{h,r}(\mathcal{D}_{(a,b]}^{h,r})) \rightarrow \mathcal{D}_{(a,b]}^{h,r}$  is an isomorphism between the corresponding subsemigroups of  $ID$ . However, it is not of the integral random mapping form unless it is the identity mapping.*

*Proof.* The isomorphism property of the inverse mapping is a consequence of the fact that  $I_{(a,b]}^{h,r}$  is an isomorphism by Theorem 1.

Now suppose that the inverse of a non-identical  $I_{(a,b]}^{h,r}$  is indeed of an integral form  $I_{(a_1,b_1]}^{h_1,r_1}$ . Then by Theorem 2

$$I_{(a,b]}^{h,r}(I_{(a_1,b_1]}^{h_1,r_1}(\nu)) = I_{(c,d]}^{s,r_2}(\nu) = \nu,$$

where

$$dr_2(t) = d(h \otimes h_1)(dr \times dr_1)(t) = d\delta_u(t) \quad \text{for some fixed } u \in (c, d].$$

In other words, for all continuous functions  $g$  on  $[c, d]$  we have

$$\int_{(a,b]} \int_{(a_1,b_1]} g(h(t)h_1(s)) dr(t) dr_1(s) = g(u).$$

Hence either  $h(t) \cdot h_1(s) = u$  (constant) for all  $t \in (a, b]$  and  $s \in (a_1, b_1]$  and  $|r(b) - r(a+)| |r_1(b_1) - r_1(a_1+)| = 1$  or  $dr \times dr_1 = \delta_t \times \delta_s$  and  $h(t)h_1(s) = u$ . Consequently, in the first case both  $h$  and  $h_1$  are constant that contradicts the assumption that  $I_{(a,b]}^{h,r}$  is non-trivial mapping. Similarly, in the second case  $r$  and  $r_1$  are Dirac measures and therefore  $I_{(a,b]}^{h,r}$  is an identity mapping. Thus this completes a proof of Theorem 3.

## 6. Fixed points (eigenfunctions) of $I_{(a,b]}^{h,r}$ .

### 6.1. Definition of fixed points

We will say that an infinitely divisible measure  $\rho$  is a *fixed point of an integral mapping*  $I_{(a,b]}^{h,r}$ , if

$$I_{(a,b]}^{h,r}(\rho) = \rho^{*c} * \delta_z, \quad \text{for some } c > 0 \text{ and } z \in E. \quad (25)$$

Equivalently, in terms of Lévy exponents, using (21)

$$\mathcal{I}_{(a,b]}^{h,r}(\Phi)(y) = c \Phi(y) + i \langle y, z \rangle \quad \text{for all } y \in E'. \quad (26)$$

**Remark 5.** i) Cf. Jurek-Vervaat (1983), Remark 5.2 to see why in the definition (23) we take  $\nu^{*c}$  instead of the more natural  $T_c \nu$  (multiplying of a corresponding a random variable by a constant).

ii) Note that (26) reads that  $\Phi$  is an eigenfunction of the mapping  $\mathcal{I}_{(a,b]}^{h,r}$  acting on the positive cone of symmetric Lévy exponents, provided we ignore the shift part.

### 6.2. Stable measures

Let us recall one of the many equivalent definitions of stable distributions. Namely, we say that  $\gamma$  is a *stable probability measures* if there exists a parameter  $0 < p \leq 2$  and for each  $t > 0$  there exists  $z(t) \in E$  such that

$$t^p \log \hat{\gamma}(y) = \log \hat{\gamma}(ty) + i \langle y, z(t) \rangle \quad \text{for all } y \in E'; \quad (27)$$

cf. Jurek (1983), Theorem 3.2. or Linde (1983). We will write  $\gamma_p \equiv \gamma$  if the above holds and say that it is a *p-stable measure*. Furthermore, we say that  $\gamma_p$  is *strictly stable*, if  $z(t) \equiv 0$  in (27).

### 6.3. Fixed points of $I_{(a,b]}^{h,r}$

**PROPOSITION 7.** *In order that  $p$ -stable measure  $\gamma_p$  be a fixed point of the mapping  $I_{(a,b]}^{h,r}$  it is necessary and sufficient that  $0 < \int_{(a,b]} |h(t)|^p |dr(t)| < \infty$ .*

*Proof.* Because of the shift  $z$  in (23) it is enough to consider only strictly stable measures. In that case we

$$\log \widehat{I_{(a,b]}^{h,r}(\gamma_p)}(y) = \int_{(a,b]} \log \hat{\gamma}_p(h(t)y) dr(t) = \int_{(a,b]} |h(t)|^p dr(t) \log \hat{\gamma}_p(y), \quad (28)$$

that is,  $p$ -stable measures  $\gamma_p$  are fixed points of  $I_{(a,b]}^{h,r}$  with  $c := \int_{(a,b]} |h(t)|^p |dr(t)|$ , which completes the proof.

Let denote by  $\mathcal{S}$  the set of all stable measures. Then we get

**COROLLARY 10.** *For the class  $\mathcal{S}$  of all stable measures*

$$[I_{(a,b]}^{h,r}(\mathcal{S}) = \mathcal{S}] \text{ iff } [0 < \int_{(a,b]} |h(t)|^p |dr(t)| < \infty, \text{ for all } 0 < p \leq 2]$$

Taking on the unit interval  $(0, 1]$  the function  $h(t) = t$  and the time change  $r(t) := t^{-\beta}$ ,  $\beta > 0$ , we see that the above corollary is not true for the mapping  $I_{(0,1]}^{t,t^{-\beta}}$  and all  $0 < p \leq 2$ .

## 7. Factorization property of measures from $\mathcal{R}_{(a,b]}^{h,r}$

### 7.1. Motivating example

Let us recall that for  $\mathbb{B}_t = (B_t^1, B_t^2)$ , Brownian motion on  $\mathbb{R}^2$ , the process

$$\mathcal{A}_t = \int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1, \quad t > 0,$$

is called *Lévy's stochastic area integral*. It is well-known that for fixed  $u > 0$ , and  $a = (\sqrt{u}, \sqrt{u}) \in \mathbb{R}^2$  we have

$$\chi(t) = E[e^{itA_u} | \mathbb{B}_u = a] = \frac{tu}{\sinh tu} \cdot \exp\{-(tu \coth tu - 1)\}, \quad t \in \mathbb{R}$$

(cf. Lévy (1951) or Yor (1992), p. 19). If  $\mu, \lambda$  and  $\nu$  are probability measures corresponding to the characteristic functions  $\chi(t), \phi(t) := tu / \sinh tu$  and  $\psi(t) := \exp[-(tu \coth tu - 1)]$ , respectively then  $\lambda = I_{(0,\infty)}^{e^{-t},t}(\nu)$ , (cf. Remark 4(c)) and moreover

$$\mu = I_{(0,\infty)}^{e^{-t},t}(\nu) * \nu \in L \equiv I_{(0,\infty)}^{e^{-t},t}(ID_{\log})$$

In other words, the conditional Levy's stochastic area integral has selfdecomposable probability distribution  $\mu$  that can be *factorized* by another selfdecomposable measure  $\lambda$  and its background driving measure  $\nu$ ; cf. Iksanov, Jurek and Schreiber (2004), p. 1367. That phenomena prompted the introduction of the notion of factorization property for the class L distributions.

### 7.2. Definition and a condition for the factorization property

If for  $\mu = I_{(a,b]}^{h,r}(\nu) \in \mathcal{R}_{(a,b]}^{h,r}$  we also have that  $I_{(a,b]}^{h,r}(\nu) * \nu \in \mathcal{R}_{(a,b]}^{h,r}$  then we say that  $\mu$  has a *factorization property*.

**PROPOSITION 8.** *Suppose that for a given functions  $h, r$  and an interval  $(a, b]$  there exist function  $h', r'$  and an interval  $(a', b']$  such that for positive measures  $\rho$  and  $\rho'$ , induced by the monotone functions  $r$  and  $r'$  respectively, we have the following*

$$\begin{aligned} h((a, b]) \cdot h'((a', b']) &= h'((a', b']) = h((a, b]) = (c, d], \text{ for some } 0 < c < d, \\ \text{and } (h \otimes h')(\rho \times \rho') &= (h\rho) - (h'\rho') \geq 0. \end{aligned} \quad (29)$$

Then  $\mathcal{D}_{(a,b]}^{h,r} \subset \mathcal{D}_{(a',b']}^{h',r'}$  and for all  $\nu \in \mathcal{D}_{(a,b]}^{h,r}$  putting  $\lambda := I_{(a',b']}^{h',r'}(\nu)$  we have

$$I_{(a',b']}^{h',r'}(I_{(a,b]}^{h,r}(\nu) * \nu) = I_{(a,b]}^{h,r}(\lambda) * \lambda = I_{(a,b]}^{h,r}(\nu).$$

In other words,  $\mathcal{R}_{(a,b]}^{h,r} = \{I_{(a,b]}^{h,r}(\lambda) * \lambda : \lambda \in I_{(a',b']}^{h',r'}(\mathcal{D}_{(a,b]}^{h,r})\}$

*Proof.* Since  $0 \leq h'\rho' \leq h\rho$  then from Proposition 4 (expressed in terms of measures) we infer the inclusion of the domains.

From (29), Theorem 2 and the additivity property from subsection 1.5. we get

$$\begin{aligned} (I_{(a,b]}^{h,\rho} \circ I_{(a',b']}^{h',\rho'})(\nu) * I_{(a',b']}^{h',\rho'}(\nu) &= I_{(a,b]}^{h,\rho}(I_{(a',b']}^{h',\rho'}(\nu)) * I_{(a',b']}^{h',\rho'}(\nu) \\ &= I_{(c,d]}^{t,(h \otimes h')(\rho \times \rho')}(\nu) * I_{(c,d]}^{t,(h'\rho')}(\nu) = I_{(c,d]}^{t,(h\rho)}(\nu) = I_{(a,b]}^{h,\rho}(\nu), \end{aligned}$$

which completes the proof.

The factorization property of a selfdecomposable measure given by the Levy's stochastic area integral is not an exception as we have

**COROLLARY 11.** *For the class L of selfdecomposable probability measures on  $E$  we have*

$$L = \{I_{(0,\infty)}^{e^{-t},t}(\nu) * \nu : \nu \in I_{(0,1]}^{s,s}(ID_{\log}(E))\}$$

*Proof.* We have that  $L = I_{(0,\infty)}^{e^{-t},t}(ID_{\log})$ ; cf. Jurek and Vervaat (1983). Then taking  $h'(s) = s, \rho' = l_1$  (Lebesgue measure on unit interval),  $a' = 0$  and  $b' = 1$  we check that conditions (29) are fulfilled. Thus Proposition 8 gives the claim in the corollary.

[The above fact was also shown in Jurek (2008), Theorem 3.1 but by a different reasoning.]

## 8. Some explicit examples.

### 8.1. Examples of domains of random integral mappings

Here we recall a few examples of domains and in some instances sketch their proofs that rely on Corollary 4.

#### Example 1.

$$\mathcal{D}_{(0,1]}^{t,-\log t} = ID_{\log}(H) := \{\mu \in ID : \int_H \log(1 + ||x||) M(dx) < \infty\}. \quad (30)$$

For this let us note that

$$\begin{aligned} \int_H (1 \wedge ||x||^2) M_{(0,1]}^{t,-\log t}(dx) &= \int_0^1 \int_H (1 \wedge t^2 ||x||^2) M(dx) \frac{dt}{t} \\ &= \int_0^1 t \int_{(||x|| \leq t^{-1})} ||x||^2 M(dx) dt + \int_0^1 \int_{(||x|| > t^{-1})} M(dx) \frac{dt}{t} = \\ &\quad 1/2 \int_H (1 \wedge ||x||^2) M(dx) + \int_{(||x|| > 1)} \log ||x|| M(dx) < \infty, \end{aligned}$$

which is equivalent with finite log-moment of  $\mu$ ; cf. Jurek and Smalara (1981) or Proposition 1.8.13 in Jurek and Mason (1993).

[Example 1 is valid on any Banach space  $E$ . However, its proof is completely different from the above for Hilbert space  $H$ ; cf. Jurek and Vervaat (1983).]

**Example 2.** (1)  $\mathcal{D}_{(0,1]}^{t,t^\beta} = ID(E)$ , for  $\beta > 0$ .

(2)  $\mathcal{D}_{(0,1]}^{t,t^\beta} = ID_\beta(H) := \{\nu \in ID(H) : \int_H ||x||^{-\beta} \nu(dx) < \infty\}$ , for  $-1 < \beta < 0$ .

(3)  $\mathcal{D}_{(0,1]}^{t,t^\beta} \cap ID^\circ = ID_\beta(H) \cap ID^\circ$ , for  $-2 < \beta \leq -1$ ; where  $ID^\circ$  denotes symmetric infinitely divisible measures.

**Remark 6.** Recall that the integral mappings  $I_{(0,1]}^{t,t^\beta}$  and their domains appeared in the context of the classes  $\mathcal{U}_\beta$  for  $-2 \leq \beta < 0$  and  $0 \leq \beta < \infty$ . The class  $\mathcal{U}_0$  coincides with the Lévy class  $L = I_{(0,\infty)}^{e^{-t},t}(ID_{\log})$ , while  $\mathcal{U}_{-2}(E)$  consists only of Gaussian measures; cf. Jurek (1988), (1989) and Jurek-Schreiber (1992).

We complete this subsection with examples of time changes given by *the incomplete Euler function*. It is defined as follows

$$\Gamma(\alpha; x) := \int_x^\infty t^{\alpha-1} e^{-t} dt, \quad \alpha \in \mathbb{R}, \quad x > 0. \quad (31)$$

For  $\alpha > 0$  the above is just the gamma function and  $\Gamma(0+) < \infty$  and thus from Proposition 3 we get

**Example 3.** For  $\alpha > 0$ ,  $\mathcal{D}_{(0,\infty)}^{t,\Gamma(\alpha;t)} = ID(E)$ .

Further for  $\alpha = 0$  we get

**Example 4.**

$$\mathcal{D}_{(0,\infty)}^{t, \int_t^\infty \frac{e^{-s}}{s} ds} = ID_{\log}(H) \quad (32)$$

Similarly as in Example 1,

$$\begin{aligned} & \int_0^\infty \int_H (1 \wedge t^2 ||x||^2) M(dx) \frac{e^{-t}}{t} dt \\ &= \int_0^\infty t \int_{(||x|| \leq t^{-1})} ||x||^2 M(dx) e^{-t} dt + \int_0^\infty \int_{(||x|| > t^{-1})} M(dx) \frac{e^{-t}}{t} dt = \\ & \int_{(||x|| > 0)} ||x||^2 \left[ \int_0^{||x||^{-1}} t e^{-t} dt \right] M(dx) + \int_{(||x|| > 0)} \left[ \int_{||x||^{-1}}^\infty \frac{e^{-t}}{t} dt \right] M(dx). \quad (33) \end{aligned}$$

Note that in the first square bracket we get

$$\int_0^{||x||^{-1}} t e^{-t} dt = 1 - e^{-||x||^{-1}} (1 + ||x||^{-1}) \leq 1 \wedge ||x||^{-2}$$

and hence the first integral in (33) is finite.

For second integral in (33) let us brake the space  $H \setminus \{0\}$  into two parts. For  $0 < ||x|| \leq 1$ ,

$$\begin{aligned} & \int_{(0 < ||x|| \leq 1)} ||x||^2 \left[ ||x||^{-2} \int_{||x||^{-1}}^\infty \frac{e^{-t}}{t} dt \right] M(dx) \\ & \leq \left[ \sup_{(a \geq 1)} a^2 \int_a^\infty \frac{e^{-t}}{t} dt \right] \int_{(||x|| \leq 1)} ||x||^2 M(dx) < \infty. \end{aligned}$$

For the part  $||x|| > 1$  we use Remark 4(a) that gives

$$\int_{||x||^{-1}}^\infty \frac{e^{-t}}{t} dt = -\mathbf{C} + \log ||x|| + \int_0^{||x||^{-1}} \frac{1 - e^{-t}}{t} dt,$$

where the integral on the right hand side is bounded by  $\int_0^1 (1-e^{-t})t^{-1}dt < \infty$ . All in all the second integral in (33) is finite if and only if  $\int_{(\|x\|>1)} \log \|x\| M(dx) < \infty$ , which completes the proof of Example 4.

**Example 5.** For  $-1 < \alpha < 0$  we have

$$\mathcal{D}_{(0,\infty)}^{t, \int_t^\infty s^{\alpha-1} e^{-s} ds} = ID_\alpha(\mathbb{R}),$$

with the notations from Example 2. For the above example and the case  $-2 < \alpha \leq -1$  cf. Sato (2006).

### 8.2. Examples of iterated integral mappings and image measures

**Example 6.** For  $\nu \in ID_{\log^m}$  and  $m = 1, 2, \dots$

$$I_{(0,\infty)}^{e^{-s}, s}(I_{(0,\infty)}^{e^{-s}, s}(\dots I_{(0,\infty)}^{e^{-s}, s}(\nu))) = I_{(0,\infty)}^{e^{-t}, \frac{t^m}{m!}}(\nu)$$

*Proof.* In view of Theorem 2 it is enough to check that for  $h(t) := e^{-t}$  and  $\rho := l$  (the Lebesgue measure on  $\mathbb{R}$ ) we have equality

$$\begin{aligned} & \int_0^\infty g(u) [(e^{-t})^{\otimes m}](l_1 \times \dots \times l_1)(du) \\ &= \int_0^\infty \int_0^\infty \dots \int_0^\infty g(e^{-(s_1+s_2+\dots+s_m)}u) ds_1 \dots ds_m = \int_0^\infty g(e^{-s}u) d\left[\frac{s^m}{m!}\right] \end{aligned}$$

for all  $g$  bounded and measurable. The first equality is just a change of variable argument. For the second, using the induction arguments, we have

$$\begin{aligned} & \int_0^\infty \left[ \int_0^\infty \dots \int_0^\infty g(e^{-(s_1+s_2+\dots+s_{m-1})}e^{-s_m}u) ds_1 \dots ds_{m-1} \right] ds_m = \\ & \int_0^\infty \int_0^\infty g(e^{-t}e^{-s_m}u) d\left[\frac{t^{m-1}}{(m-1)!}\right] ds_m = \int_0^\infty \int_{s_m}^\infty g(e^{-w}u) \frac{(w-s_m)^{m-2}}{(m-2)!} dw ds_m \\ &= \int_0^\infty g(e^{-w}u) \frac{w^{m-1}}{(m-1)!} dw = \int_0^\infty g(e^{-w}u) d\left[\frac{w^m}{m!}\right], \end{aligned}$$

which completes the proof.

**Remark 7.** The class of measures  $I_{(0,\infty)}^{e^{-t}, \frac{t^m}{m!}}(ID_{\log^m})$  coincides with the set  $L_m$  of so called *m-times selfdecomposable distributions*; cf. Jurek (2011) for the history of those classes and relevant references.

**Example 7.** For  $\beta > 0$  we have

$$I_{(0,1]}^{t^{1/\beta}, t} \circ I_{(0,1]}^{s^{1/2\beta}, s} = I_{(0,1]}^{w, 2w^\beta(1-(1/2)w^\beta)} = I_{(0,1]}^{(1-\sqrt{t})^{1/\beta}, t} \quad (34)$$

Or equivalently, for Lebeque measure  $l_1$  on the unit interval and  $0 < w \leq 1$  we get

$$(t^{1/\beta} \otimes s^{1/(2\beta)})(l_1 \times l_1)(dw) = id^{\otimes 2}(\beta t^{\beta-1} dt \times 2\beta s^{2\beta-1} ds)(dw) = 2\beta w^{\beta-1}(1-w^\beta) dw$$

*Proof.* As in Example 6, it simply follows from Theorem 2 and identity (2) because all time change functions are strictly increasing on the unit interval.

**Example 8.** For  $\beta > 0$

$$I_{(0,1]}^{t^{1/\beta}, t} \circ I_{(0,\infty)}^{e^{-s}, s} = I_{(0,\infty)}^{e^{-s}, s+\beta^{-1}e^{-\beta s}-\beta^{-1}} = I_{(0,1]}^{-w, \beta^{-1}w^\beta - \log w - \beta^{-1}}.$$

Or equivalently, for  $0 < w \leq 1$

$$(t^{1/\beta} \otimes e^{-s})(l_1 \times l)(dw) = (\beta^{-1}w^\beta - \log w - \beta^{-1})dw.$$

This we get from Theorem 2. Also cf. Czyżewska-Jankowska and Jurek (2011), Proposition 2.

**Example 9.** For  $\alpha \in \mathbb{R}$

$$I_{(0,\infty)}^{t, \Gamma(\alpha; t)} \circ I_{(0,\infty)}^{e^{-s}, s} = I_{(0,\infty)}^{t, \int_t^\infty s^{-1} \Gamma(\alpha; s) ds},$$

which follows from Theorem 2.

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